Research Article BEU Fen Bilimleri Dergisi 15(2): 647-658 (2026) DOI: 10.17798/bitlisfen.1806899

BİTLİS EREN UNIVERSITY
FEN BİLİMLERİ DERGİSİ
ISSN: 2147-3129 / e-ISSN: 2147-3188

STRONGLY STATISTICAL CONVERGENCE OF ORDER α IN METRIC SPACES

Nazlım Deniz ARAL

Bitlis Eren University, Faculty of Science Arts, Department of Mathematics, Bitlis, Türkiye, ndaral@beu.edu.tr

Article Info

Received: *October 19, 2025*
Revised: *January 6, 2026*
Accepted: *January 6, 2026*

Keywords
Statistical Convergence,
Strong Convergence,
Infinite Matrix,
Metric Space.

ABSTRACT

This paper introduces the concepts of $\mathcal{M}$-strongly statistical convergence and $[\mathcal{M}]_d^{\alpha}$-summability of order α in metric spaces (X,d), where $\mathcal{M}$ denotes a non-negative infinite matrix. The relationship between these newly defined notions and $\mathcal{M}$-strong convergence is examined in detail. Conditions ensuring the uniqueness of limits for $\mathcal{M}$ -strongly statistically convergent sequences are established. Also, we propose the definition of $\mathcal{M}$-strongly statistical Cauchy sequences of order α and study their connection to $\mathcal{M}$-strongly statistical convergence. Finally, we introduce $\mathcal{M}$-strongly statistical bounded sequences of order α in metric spaces.

1. INTRODUCTION

Since its inception in 1935, the concept of statistical convergence has drawn considerable attention in mathematical research. In 1951, the idea was formally introduced independently by Fast [12] and Steinhaus [30] as a generalization of classical (ordinary) convergence and later revisited by Schoenberg [29]. Over the years, numerous researchers have extended and developed this concept further, exploring its applications across various branches of mathematics. Notable contributions in this direction include the works of Erdös and Tenenbaum [9], Miller [23], Zygmund [33], Connor [4], Salát [28], Duman and Orhan [7], Et et al. ([10,11]), Çakallı and Savaş [5], Edely et al. [8], and Mursaleen et al. ([24-27]), among many others.

The concept of statistical convergence is fundamentally based on the asymptotic density of subsets within the set of natural numbers. Specifically, a sequence is said to be statistically convergent to a real number ζ if it converges to ζ along a subset of $\mathbb{N}$ that has asymptotic density is equal to one. Let $\mathcal{K} \subset \mathbb{N}$. If the limit exists; $D(\mathcal{K}) := \lim_{n\to\infty} \frac{1}{n} |\{\hbar \leq n : \hbar \in \mathcal{K}\}|$, then $D(\mathcal{K})$ is called the asymptotic density of the set $\mathcal{K}$, where $|.|$ denotes the cardinality of a set. Utilizing this notation of density, the statistical limit of a sequence (ζ_k) is defined as the value ζ for which the following condition holds: $\lim_{n\to\infty} \frac{1}{n} |\{\hbar \leq n: |\zeta_{\hbar} - \zeta| \geq \varepsilon\}| = 0$ for every $\varepsilon > 0$.

The collection of all such statistically convergent sequences is denoted by the set S. The notations of statistical convergence of order α for sequences of real numbers and strong p-Cesàro summability of order α were first introduced and analyzed by Çolak [3].

For a more comprehensive analysis of various generalizations and applications related to statistical convergence of order α, the reader is referred to the works in [6] and [31]. Kayan et. al. [20] introduced several notions for sequences in metric spaces, including d-statistical convergence of order α (S_d^{α}), d-statistical boundedness of order α and d-strong p-Cesàro summability of order α. For further contributions and developments related to these topics, readers may consult [1,2,21,22] and [32]. In a separate line of investigation, Karakaya and Chishti [17] proposed the concept of weighted statistical convergence in 2009. This idea was subsequently refined by Mursaleen et. al. in 2012 [27]. Later, in 2014, Ghosal [15] introduced a modification to the concept of weighted λ-statistical convergence and extended the framework defining by weighted statistical convergence of order α, incorporating the condition $\lim_{n\to\infty} inf\ p_n > 0$, where (p_n) denotes a positive weight sequence.

License: CC BY-NC 4.0

2. PRELIMINARIES

Let $\mathcal{M} = (m_{nk})$ be an infinite matrix, and let $(\zeta_n) \subset \mathbb{C}$ be a sequence with real number and $\zeta \in \mathbb{R}$. The sequence (ζ_n) is said to be $\mathcal{M}$-strongly convergent to ζ^*. If the following condition holds: $\lim_{n\to\infty} \sum_{k=1}^{\infty} m_{nk}|\zeta_k - \zeta^*| = 0$. This convergence is denoted by $\zeta_n \to \zeta^*[\mathcal{M}]$ (see [16]).

Now, consider a sequence $(T_n) \subset \mathbb{R}$ that is a non-decreasing, non-negative, and unbounded. For any subset $H \subset \mathbb{N}$, the lower α −density with respect to (T_n) is defined by

$$D_{T_n^\alpha}^{inf}(H) = \liminf_{n\to\infty} \frac{1}{T_n^\alpha} |\{k \le T_n : k \in H\}|.$$

The upper α −density of H is similarly given by

$$D_{T_n^\alpha}^{sup}(H) = \limsup_{n\to\infty} \left(\mathcal{M}\chi_H\right)_n$$

where χ_H denotes the indicator function of the set H and $\mathcal{M}$ is an averaging operator.

If both lower and upper densities agree:

$$D_{T_n^\alpha}^{inf}(H) = D_{T_n^\alpha}^{sup}(H),$$

then the set H is said to have a well-defined α −density with respect to (T_n) and the common value is denoted by $D_{T_n^\alpha}(H)$ (see [13]). In the special case where if $T_n = n$ and $\alpha = 1$, this reduces to the classical Cesàro density, written simply as $D(H)$. We say that a set $H \subset \mathbb{N}$ is dense with respect to (T_n) if $D_{T_n^\alpha}(H) = 1$, and is called an (T_n) −null set if $D_{T_n^\alpha}(H) = 0$ has density.

The notion of $\mathcal{M}$-strongly statistical convergence, which unifies and generalizes the classical concepts of statistical convergence, weighted statistical convergence, and strong convergence, has been formally introduced and studied.

Let $\mathcal{M} = (m_{nk})$ be a non-negative infinite matrix, and define row sum and cumulative sums as

$$\varrho_n = \sum_{k=1}^{\infty} m_{nk} \text{ and } T_n = \sum_{i=1}^{n} \varrho_i.$$

Suppose that the matrix $\mathcal{M}$ fulfills the conditions listed below:

(i) $\varrho_n < +\infty$ for each n $\in \mathbb{N}$,
(ii) $\lim_{n\to\infty} T_n = +\infty$.

Under these assumptions, a complex sequence $(\zeta_n) \subset \mathbb{C}$ is said to be $\mathcal{M}$-strongly statistically convergent to a limit $\zeta^* \in \mathbb{C}$ if, for every $\varepsilon > 0$, the following condition holds:

$$\lim_{n\to\infty} \frac{1}{T_n} \left| \left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki}|\zeta_i - \zeta^*| \ge \varepsilon \right\} \right| = 0.$$

We denote this convergence by $\zeta_n \overset{st}{\to} \zeta^*[\mathcal{M}]$, which indicates that the sequence (ζ_n) converges to ζ^* in the sense of $\mathcal{M}$-strong statistical convergence [18].

A matrix $\mathcal{M}$ that satisfies the conditions (i) and (ii) is referred to as an $\mathcal{T}$-type matrix [18].

Let (T_n) be a sequence that is non-decreasing, non-negative and unbounded. A non-negative sequence (ϱ_n) is said to be T_n^α-dense positive if the following condition holds:

$$\lim_{n\to\infty} sup\varrho_{v_n} > 0$$

for every subsequence $(v_n) \subset \mathbb{N}$ such that its α −density with respect to (T_n) satisfies $D_{T_n^\alpha}(v_n) = 1$. In the special case where $T_n = n$, the sequence is called dense positive but not necessarily n-dense positive [18].

3. RESULTS AND DISCUSSION

In this section, we present several fundamental properties of $\mathcal{M}$-strongly statistical convergence in the setting of general metric spaces. We also explore its relationship with strong $\mathcal{M}$-summability of order α by providing a comparative analysis. Furthermore, we establish a number of inclusion results that clarify how these two notions interact and under what conditions one implies the other.

In what follows, unless otherwise stated, we assume the (X, d) is a metric space, $0 < \alpha \leq 1$.

Definition 3.1 Let $(\zeta_n) \subset X$, $\zeta^* \in X$ and $\mathcal{M}$ is a $\mathcal{T}$-type matrix. Then, the sequence (ζ_n) is $\mathcal{M}$-strongly statistical convergence of order α to the point ζ^* if for every $\varepsilon > 0$, the following condition holds:

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon \right\} \right| = 0.$$

This type of convergence is denoted by either

$$\mathcal{M}_d^{\alpha} - \lim \zeta_n = \zeta^* \text{ or } \zeta_n \to \zeta^* (\mathcal{M}_d^{\alpha}).$$

Equivalently, this convergence condition can be expressed in terms of α −density as:

$$D_{T_n^{\alpha}} \left(\left\{ k \in \mathbb{N}: \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon \right\} \right) = 0, \qquad \forall \varepsilon > 0.$$

When $\alpha = 1$, the concept of $\mathcal{M}$-strongly statistical convergence of order α reduces to the standard notation of $\mathcal{M}$-strongly statistical convergence in a metric space introduced by Kaya [19].

Furthermore, if $\alpha = 1$, the metric is given by $d(x, y) = |x - y|$, and the matrix $\mathcal{M} = (m_{nk})$ is chosen to be the identity matrix:

$$I = (m_{nk}) = \begin{cases} 1, & n = k, \\ 0, & n \neq k \end{cases}$$

then the notation $\mathcal{M}$-strongly statistical convergence coincides with the statistical convergence as presented in [12].

Similarly, if $\alpha = 1$, $d(x, y) = |x - y|$ and the matrix $\mathcal{M} = (m_{nk})$ is defined by

$$P = (m_{nk}) = \begin{cases} p_n, & n = k, \\ 0, & n \neq k \end{cases}$$

where (p_n) is a sequence of positive weights, then $\mathcal{M}$-strongly statistical convergence becomes equivalent to the weighted statistical convergence introduced in [27].

Theorem 3.1 Let $(\zeta_n) \subset X$ with $\zeta^* \in X$ and $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. Define the quantity $\mathcal{M}_k = \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*)$. Then, the sequence (ζ_n) is said to be $\mathcal{M}$-strongly statistically convergent of order α to ζ^*, denoted by $\zeta_n \to \zeta^* (\mathcal{M}_d^{\alpha})$ if and only if there exists two non-negative sequences $(\mathcal{F}_k)$ and $(\mathcal{R}_k)$ such that $\mathcal{M}_k = \mathcal{F}_k + \mathcal{R}_k$, $\lim_{k\to\infty} \mathcal{F}_k = 0$ and $D_{T_n^{\alpha}}(\{k \in \mathbb{N} : \mathcal{R}_k \neq 0\}) = 0$.

Proof. The proof follows along the same lines as the provided in Theorem 3.1 of [18], and is therefore omitted here for brevity.

Corollary 3.1 Let $\mathcal{M}$ be a $\mathcal{T}$-type matrix, $(\zeta_n) \subset X$ and $\zeta^* \in X$. Then, $\zeta_n \to \zeta^* (\mathcal{M}) \Rightarrow \zeta_n \to \zeta^* (\mathcal{M}_d^{\alpha})$.

That is, if the sequence (ζ_n) is $\mathcal{M}$-strongly convergent to ζ^*, then it is also $\mathcal{M}$-strongly statistically convergent of order α.

However, the converse of this implication does not necessarily hold. For instance, when $\mathcal{M}$ is the identity matrix I, the implication fails (see [14]).

Corollary 3.2 Let $\mathcal{M}$ be a $\mathcal{T}$-type matrix, $(\zeta_n) \subset X$ and $\zeta^* \in X$. Then $\zeta_n \to \zeta^*(\mathcal{M}_d^\alpha)$ if and only if there exists a strictly increasing sequence of natural numbers (v_n) such that $D_{T_n^\alpha}(\{v_n: n \in \mathbb{N}\}) = 1$, and if we define a new matrix $\mathcal{F} = (\lambda_{nk}) = \left(m_{v_n k}\right)$ and $\zeta_n \to \zeta^*(\mathcal{F}_d^\alpha)$.

Theorem 3.2 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. $\mathcal{M}$-strongly statistical limit of order α is unique if and only if the sequence of row sums (ϱ_n) defined by $\varrho_n = \sum_{k=1}^{\infty} m_{nk}$ is T_n^α −dense positive, where $T_n = \sum_{i=1}^{n} \varrho_i$.

Proof. The proof follows the same argument as Theorem 4.1. in [18], and is omitted here for brevity.

Corollary 3.3 If $\mathcal{M}$ is a non-negative Toeplitz matrix, then $\mathcal{M}$-strongly statistical limit of order α is unique.

Remark 3.1 Recall the concept of weighted statistical convergence. According to Theorem 3.2, the uniqueness of the weighted statistical limit is equivalent to the condition that the weight sequence (p_n) is P_n^α-dense positive. This condition is not only necessary, but also sufficient to ensure uniqueness.

Example 3.1 Consider the matrix

$$\mathcal{M} = (m_{nk}) = \begin{cases} \frac{1}{log(n+1)} & , \text{ if } n = k,\ n \geq 20 \\ 0 & , \text{ otherwise} \end{cases}.$$

Then,

$$\varrho_n = \sum_{k=1}^{\infty} m_{nk} = \begin{cases} \frac{1}{log(n+1)} & , \text{ if } n \geq 20 \\ 0 & , \text{ if } n < 20 \end{cases},$$

and

$$T_n = \sum_{i=1}^{n} \varrho_i = \sum_{i=20}^{n} \frac{1}{log(i+1)}.$$

Note that the series $\sum_{i=20}^{\infty} \frac{1}{log(i+1)}$ diverges, so $\lim_{n\to\infty} T_n = \infty$, and $\mathcal{M}$ is a $\mathcal{T}$-type matrix. Now let (X, d) be a metric space and define the constant sequence: $(\zeta_n) = (\zeta^0, \zeta^0, \dots) \subset X$ where $\zeta^0 \in X$ is arbitrary. Fix any $\zeta^* \in X$ and $\varepsilon > 0$. Then, $d(\zeta_k, \zeta^*) = d(\zeta^0, \zeta^*)$, $m_{kk} d(\zeta_k, \zeta^*) = \frac{d(\zeta^0, \zeta^*)}{log(k+1)}$. Since $\frac{d(\zeta^0, \zeta^*)}{log(k+1)} \to 0$, the set

$$\left\{ k \leq T_n: m_{kk} \frac{d(\zeta^0, \zeta^*)}{log(k+1)} \geq \varepsilon \right\}$$

has finite for all sufficiently large n, and hence:

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} \left| \left\{ k \leq T_n: \frac{d(\zeta_k, \zeta^*)}{log(k+1)} \geq \varepsilon \right\} \right| = 0.$$

Thus, the last equation shows that every $\zeta^* \in X$ is an $\mathcal{M}$-strongly statistical limit of the constant sequence (ζ_n). Therefore, the uniqueness of the $\mathcal{M}$-strongly statistical limit fails whenever (ϱ_n) is not T_n^α −dense positive. This verifies that the dense positivity assumption in Theorem 3.2 is essential.

Theorem 3.3 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. Suppose there exists a sequence of indices $(v_n) \subset \mathbb{N}$ such that the α −density with respect to (T_n) denoted by $D_{T_n^\alpha}(v_n)$ satisfies $D_{T_n^\alpha}(v_n) = 1$. Further, assume that for every $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} m_{v_n k} = 0$ holds, and the sequence $\left(\varrho_{v_n}\right)$, where $\varrho_{v_n} = \sum_{k=1}^{\infty} m_{v_n k}$ remains bounded, i.e., $\sup_{n\in\mathbb{N}} \varrho_{v_n} < +\infty$. Under these conditions, the $\mathcal{M}$-strongly statistical convergence of order α defines a regular summability method; in other words, classical convergence $\zeta_n \to \zeta^*$ implies $\zeta_n \to \zeta^*(\mathcal{M}_d^\alpha)$.

Proof. Let $\zeta_n \to \zeta^*$ in the usual sense. Define a new matrix $\mathcal{F} = (\lambda_{nk}) = \left(m_{v_{nk}}\right)$. By the given assumptions and applying Theorem 1 from [16], it follows that $\zeta_n \to \zeta^*(\mathcal{F}_d^{\alpha})$. Then, applying Corollary 3.2, we conclude that $\zeta_n \to \zeta^*(\mathcal{M}_d^{\alpha})$.

It is worth noting that the converse does not hold in general. For instance, consider the sequence (ζ_n) defined as

$$\zeta_n = \begin{cases} 3 & , \text{ if } n = 3k^2 \\ 0 & , \text{ otherwise} \end{cases}, \ k = 1,2,3,\cdots.$$

This sequence is $\mathcal{M}$-strongly statistical limit of order α for all $\alpha > \frac{1}{2}$, yet it does not converge in the classical sense. This example demonstrates that the strong statistical convergence of order α does not guarantee classical convergence unless additional constraints are imposed.

Theorem 3.4 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix and let $0 < \alpha \leq \beta \leq 1$. Then we have $\mathcal{M}_d^{\alpha} \subseteq \mathcal{M}_d^{\beta}$ and this inclusion is strict whenever $\alpha < \frac{1}{k} < \beta$ for some $k \in \mathbb{N}$.

Proof. Suppose that the sequence (ζ_n) is $\mathcal{M}$-strongly statistically convergent of order α to a point $\zeta^* \in X$. Since $0 < \alpha \leq \beta$, for each $\varepsilon > 0$, we have

$$\frac{1}{T_n^{\beta}} \left| \left\{ k \leq T_n \ : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon \right\} \right| \leq \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n \ : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon \right\} \right|.$$

Thus, $\zeta_n \to \zeta^*$ with respect to $\mathcal{M}_d^{\beta}$ so the inclusion $\mathcal{M}_d^{\alpha} \subseteq \mathcal{M}_d^{\beta}$ is valid.

Let us consider the space $X = \mathbb{R}$ equipped with the metric $d(x, y) = |x - y|$. Take the matrix $\mathcal{M}$ as the identity matrix I so that $T_n = n$. Fix an integer $k \geq 2$ and choose parameters α and β $0 < \alpha \leq 1/k < \beta \leq 1$ and the sequence (ζ_n) is defined by

$$\zeta_n = \begin{cases} 1 & , \ if \ n = j^k, \text{for some } j \in \mathbb{N} \\ 0 & , \qquad\qquad \text{otherwise} \end{cases}.$$

Let $\zeta^* = 0$. For any $0 < \varepsilon \leq 1$, we consider the set

$$A_n(\varepsilon) = \{m \leq n : |\zeta_m - \zeta^*| \geq \varepsilon\} = \{j^k : j^k \leq n\}.$$

We have, $|A_n(\varepsilon)| = \left[\!\left[n^{1/k}\right]\!\right] \sim n^{1/k}$ as $n \to \infty$. Therefore, for $\beta > \frac{1}{k}$ $\frac{|A_n(\varepsilon)|}{T_n^{\beta}} = \frac{\left[\!\left[n^{1/k}\right]\!\right]}{n^{\beta}} \sim n^{1/k-\beta} \to 0$ $(n \to \infty)$. Hence, $(\zeta_n) \in \mathcal{M}_d^{\beta}$. However, if α is chosen $\alpha \leq \frac{1}{k}$, then $\frac{|A_n(\varepsilon)|}{T_n^{\alpha}} = \frac{\left[\!\left[n^{1/k}\right]\!\right]}{n^{\alpha}} \sim n^{1/k-\alpha} \not\to 0$. Thus, the sequence is not $\mathcal{M}$-strongly statistically convergent of order α. Thus, we conclude that $(\zeta^n) \in \mathcal{M}_d^{\beta} \backslash \mathcal{M}_d^{\alpha}$ for $\alpha \leq \frac{1}{k} < \beta$, which establishes that the inclusion is strict.

Definition 3.2 Let $\mathcal{M}$ be a $\mathcal{T}$-type matrix and let $(\zeta_n) \subset X$ be a sequence. Then, (ζ_n) is said to be an $\mathcal{M}$-strongly statistical Cauchy sequence of order α if for every $\varepsilon > 0$, there exists an index $N_{\varepsilon} \in \mathbb{N}$ such that

$$\lim_{n \to \infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n : \sum_{i=1}^{\infty} m_{ki} d\left(\zeta_i, \zeta_{N_{\varepsilon}}\right) \geq \varepsilon \right\} \right| = 0.$$

Theorem 3.5 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix, (ϱ_n) be a T_n^{α}-dense positive, $(\zeta_n) \subset X$ be a sequence and $\zeta^* \in X$. Then, every $\mathcal{M}$-strongly statistical convergent sequence in X is $\mathcal{M}$-strongly statistical Cauchy sequence.

Proof. Assume that $\mathcal{M}_d^{\alpha} - \lim \zeta_n = \zeta^*$. Then,

$$\lim_{n \to \infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \frac{\varepsilon}{2} \right\} \right| = 0$$

for each $\varepsilon > 0$. Therefore, if $\varepsilon > 0$ the equality

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) < \frac{\varepsilon}{2} \right\} \right| = 1$$

or

$$\delta_{T_n^{\alpha}} \left(\left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) < \frac{\varepsilon}{2} \right\} \right) = 1$$

holds. If we denote

$$K_{\varepsilon} = \left\{ k \in \mathbb{N} : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) < \frac{\varepsilon}{2} \right\},$$

then $D_{T_n}(K_{\varepsilon}) = 1$ and for every $k \in K_{\varepsilon}$

$$\sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) < \frac{\varepsilon}{2}.$$

Since (ϱ_n) is a bounded above, there exist $A > 0$ such that $\varrho_n \le A$ for $n \in \mathbb{N}$. On the other hand, by Proposition 4.1 in [18], ζ^* is limit of (ζ_n). From here, we say that there exists a subsequence (ζ_{k_n}) of (ζ_n) such that $\lim_{n\to\infty} \zeta_{k_n} = \zeta^*$. Definition of limit, for every $\varepsilon > 0$ there exists a $n_{\varepsilon} \in \mathbb{N}$ such that for every $n > n_{\varepsilon}$, we have

$$d(\zeta_{k_n}, \zeta^*) < \frac{\varepsilon}{2A}.$$

Let $N_{\varepsilon} = k_{n_{\varepsilon}+1}$. Then, we get

$$d(\zeta_{N_{\varepsilon}}, \zeta^*) < \frac{\varepsilon}{2A}.$$

Let $k \in K_{\varepsilon}$. So, we have

$\sum_{i=1}^{\infty} m_{ki}\, d(\zeta_i, \zeta_{N_{\varepsilon}}) \le \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) + \sum_{i=1}^{\infty} m_{ki} d(\zeta_{N_{\varepsilon}}, \zeta^*) < \frac{\varepsilon}{2} + d(\zeta_{N_{\varepsilon}}, \zeta^*) \sum_{i=1}^{\infty} m_{ki} <$
$\frac{\varepsilon}{2} + \frac{\varepsilon}{2A} A = \varepsilon.$

From here, for every $\varepsilon > 0$ there exist $N_{\varepsilon} \in \mathbb{N}$ and $K_{\varepsilon} \subset \mathbb{N}$ such that

$$\sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta_{N_{\varepsilon}}) < \varepsilon$$

for every $k \in K_{\varepsilon}$. Thus, there exists $N_{\varepsilon} \in \mathbb{N}$ for every $\varepsilon > 0$ such that

$$\delta_{T_n^{\alpha}} \left(\left\{ k \in \mathbb{N} : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta_{N_{\varepsilon}}) \ge \varepsilon \right\} \right) = 0$$

i.e, there exists $N_{\varepsilon} \in \mathbb{N}$ for every $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta_{N_{\varepsilon}}) \ge \varepsilon \right\} \right| = 0.$$

This implies that (ζ_n) is $\mathcal{M}$-strongly statistical Cauchy sequence.

Definition 3.3 Let $(\zeta_n) \subset X$ and $\mathcal{M}$ be a $\mathcal{T}$-type matrix. We say that the sequence (ζ_n) is $\mathcal{M}$-strongly statistically bounded of order α if there is a point $\zeta \in X$ and a real number $A > 0$ such that

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} |\{k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta) \geq A\}| = 0.$$

The set of all such sequences will be denoted by $B\mathcal{M}_d^{\alpha}$.

Theorem 3.6 Let (ζ_n) be a bounded sequence in X. Then the sequence (ζ_n) is $\mathcal{M}$-strongly statistically bounded of order α.

Proof. Suppose that the sequence (ζ_n) is a bounded. Then there exists a real number $A > 0$ and a point $\zeta \in X$ such that $d(\zeta_n, \zeta) < A$ for all $n \in \mathbb{N}$. Choose any $M > A$.

Now consider the expression

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta) \geq M \right\} \right| = 0.$$

Since $\{k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta) \geq M\} = \emptyset$ for each $n \in \mathbb{N}$. This completes the proof.

The converse of this theorem is not always true. For this, let us consider the metric space $X = \mathbb{R}$ with the usual metric and define the sequence (ζ_n) in the following way:

$$\zeta_n = \begin{cases} n & , \text{ if } n = k^2 \\ 0 & , \text{otherwise} \end{cases}, k = 1,2,3, \cdots.$$

The sequence is clearly unbounded. But since the inequality

$$\frac{1}{n^{\alpha}} |\{k \leq n \ : \ |\zeta_k| \geq A\}| \leq \frac{\sqrt{n}}{n^{\alpha}}$$

is satisfied for a sufficiently large $A > 0$ and the right side of the last inequality tends to 0 as $n \to \infty$ for $\frac{1}{2} < \alpha \leq 1$. We obtain that the sequence (ζ_n) is $\mathcal{M}$-strongly statistically bounded of order α.

Theorem 3.7 Let $0 < \alpha \leq \beta \leq 1$. Then,

i) Every $\mathcal{M}$-strongly statistically convergent sequence of order α is $\mathcal{M}$-strongly statistically bounded of order α. That is $\mathcal{M}_{\mathrm{d}}^{\alpha} \subseteq B\mathcal{M}_d^{\alpha}$.

ii) Every $\mathcal{M}$-strongly statistically bounded sequence of order α is also bounded of order β, that is, $B\mathcal{M}_d^{\alpha} \subseteq B\mathcal{M}_d^{\beta}$.

Proof. i) Let (ζ_n) $\mathcal{M}$-strongly statistically convergent of order α to $\zeta^* \in X$. Then, for every $\varepsilon > 0$ there exist a large enough $A > 0$ such that

$$\{k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq A\} \subset \{k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon\}.$$

Since the right-hand side has density zero in the α-order statistical sense, so does the left-hand side. This shows that $(\zeta_n) \in B\mathcal{M}_d^{\alpha}$, i.e., the sequence $\mathcal{M}$-strongly statistically bounded of order α.

ii) Assume that $(\zeta_n) \in B\mathcal{M}_d^{\alpha}$. Then, for some $A > 0$ and $\zeta \in X$, we have

$$\lim_{n\to\infty} \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta) \geq A \right\} \right| = 0.$$

Now, since $\alpha \leq \beta$,

$$\frac{1}{T_n^{\beta}} \left| \left\{ k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta) \geq A \right\} \right| \leq \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta) \geq A \right\} \right|.$$

As the right-hand side tends to 0, so does the left hand side. Hence, $(\zeta_n) \in B\mathcal{M}_d^{\beta}$, completing the proof.

Theorem 3.8 Let $(\zeta_n) \subset X$ and let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. If (ζ_n) is a $\mathcal{M}$-strongly statistical Cauchy sequence of order α, then it is $\mathcal{M}$-strongly statistically bounded of order α.

Proof. Assume that (ζ_n) is a $\mathcal{M}$-strongly statistical Cauchy sequence of order α. Then, for every $\varepsilon > 0$, there exist an index $N_\varepsilon \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} \left| \left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki} d\left(\zeta_i, \zeta_{N_\varepsilon}\right) \ge \varepsilon \right\} \right| = 0.$$

For any sufficiently large number $A > 0$, we have the following inclusion

$$\left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki} d\left(\zeta_i, \zeta_{N_\varepsilon}\right) \ge A \right\} \subset \left\{ k \le T_n : \sum_{i=1}^{\infty} m_{ki} d\left(\zeta_i, \zeta_{N_\varepsilon}\right) \ge \varepsilon \right\}.$$

Therefore, (ζ_n) is $\mathcal{M}$-strongly statistically bounded of order α.

Definition 3.4 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. A sequence $(\zeta_n) \subset X$ is said to be $[\mathcal{M}]_d^\alpha$-summable (or $\mathcal{M}$-strongly d-summable of order α) to a point $\zeta^* \in X$, if

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} \sum_{k=1}^{T_n} \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) = 0.$$

We denote this by $[\mathcal{M}]_d^\alpha - lim\zeta_n = \zeta^*$ or $\zeta_n \to \zeta^*([\mathcal{M}]_d^\alpha)$. The set of all sequences $[\mathcal{M}]_d^\alpha$-summable will be denoted by $[\mathcal{M}]_d^\alpha$.

Theorem 3.9 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix and $0 < \alpha \le \beta \le 1$. If a sequence $(\zeta_n) \subset X$ is $[\mathcal{M}]_d^\alpha$-summable to $\zeta^* \in X$, then it is also $\mathcal{M}_d^\beta$-statistically convergent to ζ^*. However, the converse does not hold in general.

Proof. This implication is straightforward since $\alpha \le \beta$ implies that $\frac{1}{T_n^\beta} \le \frac{1}{T_n^\alpha}$ for sufficiently large n, and so omitted.

For the converse, let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define a sequence (ζ_n) by

$$(\zeta_n) = \begin{cases} 3, & n = k^2, \\ 0, & n \ne k^2. \end{cases}$$

Let us show that (ζ_n) is $\mathcal{M}_d^\beta$-statistically convergent to 0 for $\beta > \frac{1}{2}$. For every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{T_n^\beta} \left| \left\{ k \le T_n \ : \ \sum_{i=1}^{\infty} m_{ki} |\zeta_i - 0| \ge \varepsilon \right\} \right| \le \frac{\sqrt{n}}{n^\beta} \to 0,$$

as $n \to \infty$. So we have $\zeta_n \to 0(\mathcal{M}_d^\beta)$. However, consider the strong summability of order $\alpha < \frac{1}{2}$. Then,

$$\lim_{n\to\infty} \frac{1}{T_n^\alpha} \sum_{k=1}^{T_n} \sum_{i=1}^{\infty} m_{ki} |\zeta_i| \ge \frac{3\sqrt{n}}{n^\alpha} \to \infty$$

as $n \to \infty$. Hence, the sequence is not $[\mathcal{M}]_d^\alpha$-summable.

Corollary 3.4 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. If a sequence $(\zeta_n) \subset X$ is $[\mathcal{M}]_d^\alpha$-summable to $\zeta^* \in X$, then it is also $\mathcal{M}_d^\alpha$-statistically convergent to ζ^*. However, the converse does not necessarily hold.

Theorem 3.10 Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix, $\alpha = 1$ and let (ζ_n) is a bounded sequence in a metric space. If (ζ_n) is $\mathcal{M}_d$-statistically convergent to $\zeta^* \in X$, then it is $[\mathcal{M}]_d$-summable to ζ^*.

Proof. Suppose that (ζ_n) is $\mathcal{M}_d$-statistically convergent to ζ^*. Then for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{T_n} \left| \left\{ k \le T_n \ : \ \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \ge \varepsilon \right\} \right| = 0.$$

Since (ζ_n) is a bounded sequence in a metric space X, there exists a constant $A > 0$ such that $d(\zeta_i, \zeta^*) \leq A$ for all $i \in \mathbb{N}$. Therefore, we have

$$\sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \leq A \sum_{i=1}^{\infty} m_{ki} = A\varrho_k.$$

Hence, for every $k \leq T_n$, $A\varrho_k \leq AD$, where $D = \sup_{k \leq T_n} \varrho_k$. Now, we write the mean of the summation

$$\frac{1}{T_n}\sum_{k=1}^{T_n}\sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) = \frac{1}{T_n}\sum_{k\in C_n}\sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) + \frac{1}{T_n}\sum_{k\in H_n}\sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*)$$

where $C_n = \{k \leq T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) < \varepsilon\}$, $H_n = \{k \leq T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon\}$.

We conclude that,

$$\lim_{n\to\infty} \frac{1}{T_n} \sum_{k=1}^{T_n} \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) = 0$$

which shows that (ζ_n) is $[\mathcal{M}]_d$-summable to ζ^*.

Theorem 3.11. Let $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix. If $0 < \alpha \leq \beta \leq 1$, then $[\mathcal{M}]_d^{\alpha} \subset [\mathcal{M}]_d^{\beta}$.

Proof. Thus, the inclusion is obtained without need of a detailed proof.

Theorem 3.12. $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix and α be a real number such that $0 < \alpha \leq 1$. Then the following statements hold;

- **i)** If $\liminf_n \frac{T_{n+1}}{T_n} > 1$ and $\limsup_n \varrho_n < c$, then $S^{d,\alpha} \subseteq \mathcal{M}_d^{\alpha}$.
- **ii)** Let $\lim_n \frac{1}{T_n^{\alpha}} \sum_{i=1}^{\infty} m_{n+1,i} = 0$. If $\liminf_n \varrho_n > c > 0$, then $\mathcal{M}_d^{\alpha} \subseteq S^{d,\alpha}$.

Proof.

i) Assume $\liminf_m \frac{T_{m+1}}{T_m} > 1$. Then there exists a $c > 0$ such that $\frac{T_{m+1}}{T_m} \geq 1 + c$ for sufficiently large m, which implies $\frac{1}{T_{m+1}^{\alpha}} \leq \frac{1}{(1+c)^{\alpha} T_m^{\alpha}}$. Now let n be a sufficiently large number, then exists a positive integer m such that $T_m < n \leq T_{m+1}$. Assume $\zeta_n \to \zeta^*(S_d^{\alpha})$ for every $\varepsilon > 0$ and sufficiently large n, we have

$$\lim_{n\to\infty} \frac{1}{T_{m+1}^{\alpha}} \left| \left\{ k \leq T_m : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon \right\} \right| \leq \frac{1}{T_{m+1}^{\alpha}} |\{k \leq T_m \ : \ d(\zeta_i, \zeta^*) \geq \varepsilon\}|$$

$$\leq \frac{1}{(1+c)^{\alpha} T_m^{\alpha}} |\{k \leq T_m \ : \ d(\zeta_i, \zeta^*) \geq \varepsilon\}|.$$

So it follows that $\zeta_n \to \zeta^*(\mathcal{M}_d^{\alpha})$.

ii) Assume $\zeta_n \to \zeta^*(\mathcal{M}_d^{\alpha})$. Since $\varrho_n \geq c > 0$, we get

$$\left\{ k \leq T_n : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon \right\} \subseteq \left\{ k \leq T_n \ : \ d(\zeta_i, \zeta^*) \geq \frac{\varepsilon}{c} \right\}.$$

Then for every $\varepsilon > 0$, we can write

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{k \leq n : \ d(\zeta_i, \zeta^*) \geq \varepsilon\}| \leq \frac{1}{T_n^{\alpha}} \left| \left\{ k \leq T_n \ : \ d(\zeta_i, \zeta^*) \geq \frac{\varepsilon}{c} \right\} \right|$$

$$+ \frac{1}{T_n^{\alpha}} \sum_{i=1}^{\infty} m_{n+1,i}.$$

Since $\zeta_n \to \zeta^*(\mathcal{M}_d^{\alpha})$, it follows that $\zeta_n \to \zeta^*(S_d^{\alpha})$.

Theorem 3.13. $\mathcal{M} = (m_{nk})$ be a $\mathcal{T}$-type matrix and α and β be two real numbers such that $0 < \alpha \leq \beta \leq 1$. If $\frac{T_n^\alpha}{T_{n+1}^\beta} = \vartheta > 0$ and $\limsup_n \varrho_n < c$ then $S^{d,\alpha} \subseteq \mathcal{M}_d^\beta$.

Proof. From the Theorem 3.12, we have

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : d(\zeta_i, \zeta^*) \geq \varepsilon\}| = 0.$$

Let $T_m < n \leq T_{m+1}$. Then

$$\frac{1}{T_{m+1}^\alpha} |\{k \leq T_m : d(\zeta_k, \zeta^*) \geq \varepsilon\}| \leq \frac{1}{n^\alpha} |\{k \leq n : d(\zeta_k, \zeta^*) \geq \varepsilon\}| \to 0.$$

Since $\varrho_n < c$, we have

$$\left\{k \leq T_m : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon\right\} \subseteq \left\{k \leq T_m : d(\zeta_k, \zeta^*) \geq \frac{\varepsilon}{c}\right\}.$$

Thus

$$\frac{1}{T_{m+1}^\beta} |\{k \leq T_m : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon\}| \leq \frac{T_m^\alpha}{T_{m+1}^\beta} \frac{1}{T_m^\alpha} \left|\left\{k \leq T_m : d(\zeta_k, \zeta^*) \geq \frac{\varepsilon}{c}\right\}\right|.$$

Since $\frac{T_n^\alpha}{T_{n+1}^\beta} \geq \vartheta$,

$$\lim_{m\to\infty} \frac{1}{T_{m+1}^\beta} |\{k \leq T_m : \sum_{i=1}^{\infty} m_{ki} d(\zeta_i, \zeta^*) \geq \varepsilon\}| = 0.$$

Therefore, $\zeta_n \to \zeta^*(\mathcal{M}_d^\alpha)$, and so $S^{d,\alpha} \subseteq \mathcal{M}_d^\beta$.

4. CONCLUSION and SUGGESTIONS

We were motivated to write this paper in light of Kayan, Çolak and Altın, the articles that followed these studies. In this paper, we introduced $\mathcal{M}$-strongly statistical convergence of order α, $[\mathcal{M}]_d^\alpha$-summability of order α in a metric space and investigated the relations of essential notions. Then, we proposed the statistical Cauchy sequences. In addition, investigated $\mathcal{M}$-strongly statistical bounded sequences of order α in a metric space. In future studies, concepts such as ideal convergence, double sequences and other generalizations of statistical convergence can be analyzed in the space in question. Besides, with the help of this study, the relationship between statistical convergence and summability theory can be discussed.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the author.

REFERENCES

[1] N. D. Aral, H. S. Kandemir and M. Et, "Deferred d-Statistical Boundedness of Order α in Metric Spaces," *Dera Natung Government College Research Journal*, vol. *9*, no.1, pp.1-12, 2024.

[2] B. Bilalov and T. Nazarova, "On statistical convergence in metric space," *Journal of Mathematics Research*, vol.7, pp. 37–43, 2015.

[3] R. Çolak, "Statistical convergence of order α," *Modern Methods in Analysis and Its Applications*, *Anamaya Pub.*, New Delhi, India, pp. 121-129, 2010.

[4] J. S. Connor, "The statistical and strong p-Cesaro convergence of sequences," *Analysis*, vol.8, pp. 47-63, 1988.

[5] H. Çakallı and E. Savaş, "Statistical convergence of double sequences in topological group," *J. Comput. Anal. Appl.*, vol.12, pp. 421–426, 2010.

[6] P. Das, S. Ghosal and S. Som, "Statistical convergence of order α in probability", *Arab J. Math. Sci*., vol.21, no.2, pp. 253–265, 2015.

[7] O. Duman and C. Orhan, "μ-statistically convergent function sequences," *Czechoslovak Math. J.*, vol.54, pp. 413-422, 2004.

[8] O. H. H. Edely, S. A. Mohiuddine and A. K. Noman, "Korovkin type approximation theorems obtained through generalized statistical convergence," *Appl. Math. Lett*., vol.23, pp. 1382–1387, 2010.

[9] P. Erdös and G. Tenenbaum, "Sur les densités de certaines soites d'entires," *Proceeding of the London Math. Socaity*, vol.59, no.3, pp. 417-438, 1989.

[10] M. Et, H. Altınok and R. Çolak, "On λ-statistical convergence of difference sequences of fuzzy numbers," *Inform. Sci*., vol.176, pp. 2268–2278, 2006.

[11] M. Et, M. Çınar, H. Şengül Kandemir, "Deferred statistical convergence of order α in metric spaces," *Aims Mathematics*, vol.5, no.4, pp. 3731–3740, 2020.

[12] H. Fast, "Sur la convergence statistique," *Colloq. Math*., vol.2, pp. 241-244, 1951.

[13] A. R. Freedman and J. J. Sember, "Density and summability," *Pacific J. Math.*, vol. 95, no. 2, pp. 293–305, 1981.

[14] J. A. Fridy, "On statistical convergence," *Analysis (Berlin)*, vol. 5, no. 4, pp. 301–313, 1985.

[15] S. Ghosal, "Weighted statistical convergence of order α and its applications," *J. Egyptian Math. Soc.*, vol. 24, pp. 60–67, 2016.

[16] H. J. Hamilton, J. D. Hill, "On strong summability," *Am. J. Math*., vol.60, no.3, pp. 588-594, 1938.

[17] V. Karakaya and T. A. Chishti, "Weighted statistical convergence," *Iran J. Sci. Technol. Trans. A Sci.*, vol.33, no. 3, pp. 219–223, 2009

[18] U. Kaya and N. D. Aral, "Strongly Statistical Convergence," *Ukr. Math. J.*, vol.72, no. 2, pp. 246–259, 2020.

[19] İ. Kaya, "Metrik Uzaylarda A-güçlü İstatistiksel Yakınsaklık" *Bitlis Eren University Graduate School of Natural and Applied Sciences, Master Thesis*, 2017.

[20] E. Kayan, R. Çolak and Y. Altın, "d-statistical convergence of order α and d-statistical boundedness of order α in metric spaces," *U.P.B. Sci. Bull., Series A*, vol.180, no. 4, pp. 229–238, 2018.

[21] Ö. Kişi, S. Çetin, and M. Gürdal, "Deferred Statistical Convergence of Sequences in Octonion-valued Metric Spaces," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 74, no. 3, pp. 375–394, Apr. 2025.

[22] M. Küçükaslan, U. Değer and O. Dovgoshey, "On statistical convergence of Metric-Valued Sequences," *Ukr. Math. J.,* vol.66, no. 5, pp. 712-720, 2014.

[23] H. I. Miller, "A measure theoretical subsequence characterization of statistical convergence," *Trans. of the Amer. Math. Soc*., vol.447, no. 5, pp. 1811-1819, 1995.

[24] M. Mursaleen, "λ-statistical convergence," *Math. Slovaca*, vol.50, pp. 111–115, 2000.

[25] M. Mursaleen, C. Çakan, S. A. Mohiuddine and E. Savaş, "Generalized statistical convergence and statistical core of double sequences," *Acta Math. Sin.* (Engl. Ser.), vol.26, pp.2131–2144, 2010.

[26] M. Mursaleen and O. H. H. Edely, "Statistical convergence of double sequences," *J. Math. Anal.Appl.*, vol. 288, pp. 223–233, 2003.

[27] M. Mursaleen, V. Karakaya, M. Ertürk and F. Gürsoy, "Weighted statistical convergence and its application to Korovkin type approximation theorem," *Appl. Math. Comput.*, vol. 218, pp. 9132–9137, 2012.

[28] T. Salat, "On statistically convergent sequences of real numbers," *Mat. Slovaca*, vol. 30, pp. 139–150, 1980.

[29] I. J. Schoenberg, "The Integrability of certain functions and related summability methods," *Amer. Math. Monthly*, vol. 66, pp. 361-375, 1959.

[30] H. Steinhaus, "Sur la convergence ordinaire et la convergence asymptotique," *Colloq. Math.*, vol. 2, pp. 71-74, 1951.

[31] H. Şengül and M. Et, "On lacunary statistical convergence of order α," *Acta Math. Sci*., vol. 2, no. 34, pp. 473–482, 2014.

[32] H. Şengül, M. Et and H. Çakallı, "Lacunary d statistical convergence and lacunary d statistical boundedness in metric spaces," *AIP Conf. Proc.*, 6 December 2019; 2183 (1)

[33] A. Zygmund, *Trigonometric Series*, Cam. Uni. Press, Cambridge, UK., 1979.